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AN ANALYSIS OF EXACT AND APPROXIMATE EQUATIONS FOR THE TEMPERATURE DISTRIBUTION IN AN INSULATED THICK SKIN SUBJECTED TO AERODYNAMIC HEATING

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ERRATA

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The following corrections should be made to equations (17b) and (19a) of the subject paper.

Page 11, last line: In equation (17b), the exponential $e^{-\gamma_n \alpha_2}$ should be $e^{-\gamma_n^2 \alpha_2}$.

Page 12: In the second line of equation (19a), the exponential $e^{-\gamma_n \alpha_2}$ should be $e^{-\gamma_n^2 \alpha_2}$; also, in the third line $e^{-\gamma_n(\alpha_2 - u)}$ should be $e^{-\gamma_n^2(\alpha_2 - u)}$.

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SUMMARY

The problem of calculating the temperature distribution in an insulated slab is investigated. Exact and approximate solutions are obtained, and the results are compared to determine the ranges of applicability of the approximations. The approximations are found to be within 5 percent of the exact solution when the ratio of the thermal capacitance of the metal to that of the insulation and the ratio of the conductance of the metal to that of the insulation are sufficiently large. The roots of the characteristic equation of the exact solution are generally applicable to the two-slab heat-transfer problem and are tabulated up to the first nine roots.

INTRODUCTION

Some aircraft designs incorporate insulation to protect the structure from the effects of the elevated temperatures encountered in high-speed flight. (See refs. 1 and 2.) The efficient design of such structures requires that the temperatures of the insulation and metal skin be known. This temperature-distribution problem is the same as that of the boundary-value problem of two slabs of different thermal characteristics joined together at one face. Application of the exact solution involves considerable computational effort, and when great accuracy is not required, simplified approximate expressions are desired.

The exact solution and two approximate solutions are presented for the insulated slab subjected to a step-function adiabatic wall temperature. The solutions to problems involving variable adiabatic wall temperatures can be obtained from these results by applying Duhamel's integral. The characteristic equation is independent of the adiabatic-wall-temperature variation; the first nine roots of this equation are tabulated for a wide range of thermal properties of the slabs for reference purposes.

The results of the step-function temperature problem are shown graphically to facilitate comparison with the approximate solutions. The charts may be used to evaluate the accuracy of approximate solutions for particular numerical problems.

SYMBOLS

A_k, B_k	arbitrary constants
c	specific heat
F	geometric factor
$G(s)$	expression defined in the text (eq. (10))
h	aerodynamic heat-transfer coefficient
k	thermal conductivity
$p(s)$	function of (s)
q	thermal flux density
$q(s)$	function of (s)
s	Laplace parameter
s_n	values of s that are poles of transformed solution (see eq. (12))
T	temperature
t	thickness
u	dummy variable of integration
w	weight density
x,y,z	body coordinates

$$\alpha_i = \frac{k_i \tau}{c_i w_i t_i^2}$$

$\beta^2 = \frac{\alpha_1}{\alpha_2} = \frac{k_1}{k_2} \frac{c_2 w_2 t_2{}^2}{c_1 w_1 t_1{}^2}$

γ_n roots of characteristic equation

$\zeta = \frac{k_1}{h t_1}$

$\eta_i = \frac{z_i}{t_i}$

θ_i Laplace transform of T_i

λ nondimensional time parameter associated with the first approximate solution, $\frac{k_1 \tau}{c_2 w_2 t_2 t_1}$

$\xi = \frac{k_2 t_1}{k_1 t_2}$

$\rho = \frac{c_2 w_2 t_2}{c_1 w_1 t_1}$

σ Stefan-Boltzmann constant

τ time

ϕ_j expressions defined in the text (eq. (37a))

ω nondimensional time parameter associated with the second approximate solution, $\frac{\lambda}{(1 + \zeta)\left(1 + \frac{1}{2\rho}\right)}$

Subscripts:

o initial

1 refers to insulation

2 refers to metal plate

aw adiabatic wall

eq equilibrium

fs free space

i indicial notation for metal slab or insulating surface

n indicial notation in expansions

ANALYSIS

The problem of determining the temperature distribution in an aircraft structure insulated from the effects of aerodynamic heating is analyzed by considering the structure to be two slabs of different material joined at one face. The exact solution is obtained in classical form by assuming that the slabs are heated uniformly over one of the exposed surfaces. The entry of heat at the heated surface is governed by Newton's "law of cooling," which applies when a solid surface is in an atmosphere at a temperature different from that of the surface. Figure 1(a) is a schematic diagram of the system. The atmospheric temperature is taken as the adiabatic-wall temperature associated with aerodynamic heating, denoted by T_{aw}; this temperature is the effective air temperature for heat-transfer purposes. The thermal conductance of the air boundary layer adjacent to the insulation surface is denoted by h.

In all cases of the analysis h has been assumed constant, even though T_{aw} may be a function of time. It also has been assumed that no heat is conducted away from the unheated slab surface and, therefore, that all heat that has entered the slabs is stored therein. All material thermal properties are assumed constant.

Approximate solutions are obtained by reducing the partial-differential heat-transfer equations to ordinary differential equations by assuming a linear temperature gradient through the insulation and no gradient through the metal structure. When the heating rate is low, and when most of the thermal capacity of the system is located in the metal structure, the problem approaches that of a steady state in which the temperature gradient through the insulation would be linear. Also, at low heating rates, the temperature gradient through the back (metal) slab is small. Results identical to those obtained from the reduced differential equations may be obtained by applying the same approximations to the exact solution.

Exact Solution

Constant adiabatic wall temperature.- Figure 1(a) is a sketch of the insulated slab showing the coordinate systems that are used. The flow of heat in each section obeys the standard equation for heat conduction:

$$c_i w_i \frac{\partial T_i}{\partial \tau} = k_i\left(\frac{\partial^2 T_i}{\partial {x_i}^2} + \frac{\partial^2 T_i}{\partial {y_i}^2} + \frac{\partial^2 T_i}{\partial {z_i}^2}\right) \qquad (i = 1,\ 2)$$

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The heating is assumed to be uniform over the surface under consideration; the equation then reduces to the one-dimensional form

$$c_i w_i \frac{\partial T_i}{\partial \tau} = k_i \frac{\partial^2 T_i}{\partial {z_i}^2} \qquad (i = 1,\ 2) \qquad (1)$$

where it is further assumed that the material properties are independent of time and temperature.

Equation (1) may be nondimensionalized, for convenience, by setting

$$\alpha_i = \frac{k_i \tau}{c_i w_i {t_i}^2} \qquad (2a)$$

and

$$\eta_i = \frac{z_i}{t_i} \qquad (2b)$$

These substitutions reduce equation (1) to

$$\frac{\partial T_i}{\partial \alpha_i} = \frac{\partial^2 T_i}{\partial {\eta_i}^2} \qquad (i = 1,\ 2) \qquad (3)$$

The nondimensional boundary conditions are

$$\frac{k_1}{t_1} \frac{\partial T_1}{\partial \eta_1}\left(1,\alpha_1\right) = h\left[T_{aw} - T_1\left(1,\alpha_1\right)\right] \qquad (4a)$$

$$\frac{k_1}{t_1}\frac{\partial T_1}{\partial \eta_1}(0,\alpha_1) = \frac{k_2}{t_2}\frac{\partial T_2}{\partial \eta_2}(1,\alpha_2) \tag{4b}$$

$$T_1(0,\alpha_1) = T_2(1,\alpha_2) \tag{4c}$$

$$\frac{k_2}{t_2}\frac{\partial T_2}{\partial \eta_2}(0,\alpha_2) = 0 \tag{4d}$$

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Boundary condition (4a) states that the heat enters the insulation surface at a rate proportional to the difference between the adiabatic wall temperature and the temperature of the outer surface of the insulation. Equation (4b) states that, at the interface, all heat leaving the first slab enters the second slab. Equation (4c) equates the temperature of the first slab to that of the second slab at the interface, and equation (4d) states that no heat is lost through the back surface of the second slab.

The initial condition is

$$T_i(\eta_i,0) = T_o$$

In order to obtain a common time variable, the notation

$$\beta^2 = \frac{\alpha_1}{\alpha_2} \tag{5}$$

is adopted. Equations (3) may then be written

$$\frac{\partial T_1}{\partial \alpha_2} = \beta^2 \frac{\partial^2 T_1}{\partial {\eta_1}^2} \tag{6a}$$

$$\frac{\partial T_2}{\partial \alpha_2} = \frac{\partial^2 T_2}{\partial {\eta_2}^2} \tag{6b}$$

Equation (4a) is a nonhomogeneous boundary condition. The solution to the problem might be accomplished by redefining the dependent variable to eliminate the nonhomogeneity at the boundaries. However, such a transformation is unnecessary if the solution is obtained by means of the Laplace transform technique.

The Laplace transforms of equations (6a) and (6b) are:

$$s\theta_1 - T_o = \beta^2 \frac{\partial^2 \theta_1}{\partial \eta_1^{\,2}} \tag{7a}$$

$$s\theta_2 - T_o = \frac{\partial^2 \theta_2}{\partial \eta_2^{\,2}} \tag{7b}$$

The transformation of equations (4) yields

$$\frac{k_1}{t_1}\frac{\partial \theta_1}{\partial \eta_1}(1,s) = h\left[\frac{T_{aw}}{s} - \theta_1(1,s)\right] \tag{8a}$$

$$\frac{k_1}{t_1}\frac{\partial \theta_1}{\partial \eta_1}(0,s) = \frac{k_2}{t_2}\frac{\partial \theta_2}{\partial \eta_2}(1,s) \tag{8b}$$

$$\theta_1(0,s) = \theta_2(1,s) \tag{8c}$$

$$\frac{k_2}{t_2}\frac{\partial \theta_2}{\partial \eta_2}(0,s) = 0 \tag{8d}$$

where T_{aw} is assumed to be a constant for the step-function solutions.

Solutions to equations (7) are

$$\theta_1 = A_1 \sinh \frac{\sqrt{s}}{\beta}\eta_1 + A_2 \cosh \frac{\sqrt{s}}{\beta}\eta_1 + \frac{T_o}{s} \tag{9a}$$

$$\theta_2 = B_1 \sinh \sqrt{s}\,\eta_2 + B_2 \cosh \sqrt{s}\,\eta_2 + \frac{T_o}{s} \tag{9b}$$

where A_1, A_2, B_1, and B_2 are arbitrary constants to be determined by applying boundary conditions (8).

Applying boundary condition (8d) gives

$$\frac{k_2}{t_2}\sqrt{s}\,B_1 = 0$$

Therefore $B_1 = 0$, since, in general, k_2/t_2 or $\sqrt{s}$ are not zero. From condition (8c)

$$A_2 + \frac{T_o}{s} = B_2 \cosh \sqrt{s} + \frac{T_o}{s}$$

$$A_2 = B_2 \cosh \sqrt{s}$$

Applying condition (8b) gives

$$\frac{k_1}{t_1}\frac{\sqrt{s}}{\beta} A_1 = \frac{k_2}{t_2}\sqrt{s}\,B_2 \sinh \sqrt{s}$$

$$A_1 = \beta \frac{k_2 t_1}{k_1 t_2} B_2 \sinh \sqrt{s}$$

Applying condition (8a) yields

$$\frac{k_1}{t_1}\frac{\sqrt{s}}{\beta}\left(A_1 \cosh \frac{\sqrt{s}}{\beta} + A_2 \sinh \frac{\sqrt{s}}{\beta}\right) = h\left(\frac{T_{aw}}{s} - \frac{T_o}{s} - A_1 \sinh \frac{\sqrt{s}}{\beta} - A_2 \cosh \frac{\sqrt{s}}{\beta}\right)$$

or

$$B_2\left[\beta \frac{k_2 t_1}{k_1 t_2} \sinh \sqrt{s}\left(\frac{k_1}{ht_1}\frac{\sqrt{s}}{\beta} \cosh \frac{\sqrt{s}}{\beta} + \sinh \frac{\sqrt{s}}{\beta}\right)\right.$$

$$\left. + \cosh \sqrt{s}\left(\frac{k_1}{ht_1}\frac{\sqrt{s}}{\beta} \sinh \frac{\sqrt{s}}{\beta} + \cosh \frac{\sqrt{s}}{\beta}\right)\right] = \frac{T_{aw} - T_o}{s}$$

This result may be written as

$$B_2 = \frac{T_{aw} - T_o}{sG(s)}$$

where

$$G(s) = \beta \frac{k_2 t_1}{k_1 t_2} \sinh \sqrt{s} \left(\frac{k_1}{ht_1} \frac{\sqrt{s}}{\beta} \cosh \frac{\sqrt{s}}{\beta} + \sinh \frac{\sqrt{s}}{\beta} \right)$$

$$+ \cosh \sqrt{s} \left(\frac{k_1}{ht_1} \frac{\sqrt{s}}{\beta} \sinh \frac{\sqrt{s}}{\beta} + \cosh \frac{\sqrt{s}}{\beta} \right) \tag{10}$$

Substitution of the preceding values for the arbitrary constants A_k and B_k into equations (9) yields

$$\theta_1 = \beta\xi \left(\frac{T_{aw} - T_o}{sG(s)} \right) \sinh \sqrt{s} \sinh \frac{\sqrt{s}}{\beta} \eta_1 + \left(\frac{T_{aw} - T_o}{sG(s)} \right) \cosh \sqrt{s} \cosh \frac{\sqrt{s}}{\beta} \eta_1 + \frac{T_o}{s} \tag{11a}$$

$$\theta_2 = \left(\frac{T_{aw} - T_o}{sG(s)} \right) \cosh \sqrt{s} \eta_2 + \frac{T_o}{s} \tag{11b}$$

where

$$\xi = \frac{k_2 t_1}{k_1 t_2}$$

The inverse transforms of equations (11) can be found formally by the method given in reference 3 (p. 170)

$$T(\tau) = L^{-1}\left[\frac{p(s)}{q(s)} \right] = \sum_{n=1}^{\infty} \frac{p(s_n)}{q'(s_n)} e^{s_n \tau} \tag{12}$$

provided the numerators and denominators are analytic, all of the poles are contained in $q(s_n)$, and $p(s_n) \neq 0$. The prime on q indicates differentiation with respect to s. It can be shown by expanding the functions of equations (11) that these conditions are met. Therefore, after the indicated differentiation has been performed, terms collected, and ζ substituted for k_1/ht_1, this transform becomes

$$\begin{aligned} q'(s) &= sG'(s) + G(s) \\ &= \frac{\sqrt{s}}{2}\left\{\cosh \sqrt{s}\left[\xi\zeta \sqrt{s} \cosh \frac{\sqrt{s}}{\beta} + \beta\xi \sinh \frac{\sqrt{s}}{\beta}\right.\right. \\ &\quad \left. + \frac{\zeta}{\beta^2}\left(\beta \sinh \frac{\sqrt{s}}{\beta} + \sqrt{s} \cosh \frac{\sqrt{s}}{\beta}\right) + \frac{1}{\beta} \sinh \frac{\sqrt{s}}{\beta}\right] \\ &\quad + \sinh \sqrt{s}\left[\frac{\zeta\sqrt{s}}{\beta} \sinh \frac{\sqrt{s}}{\beta} + \cosh \frac{\sqrt{s}}{\beta} + \frac{\xi\zeta}{\beta}\left(\beta \cosh \frac{\sqrt{s}}{\beta}\right.\right. \\ &\quad \left.\left.\left. + \sqrt{s} \sinh \frac{\sqrt{s}}{\beta}\right) + \xi \cosh \frac{\sqrt{s}}{\beta}\right]\right\} + G(s) \end{aligned} \tag{13}$$

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For all the poles except $s_n = 0$, it should be noted that $G(s) = 0$. At the pole $s_n = 0$, the inverse transform is

$$T_1\Big|_{s_n=0} = T_{aw} \tag{14a}$$

and

$$T_2\Big|_{s_n=0} = T_{aw} \tag{14b}$$

Equations (14) are the steady-state solutions to the problem.

The transient portion of the solution is obtained when $G(s) = 0$. Equation (10) can be put into the form of real functions by making the substitution

$$s_n = -\gamma_n^2$$

or

$$\sqrt{s_n} = i\gamma_n \tag{15}$$

to yield

$$\begin{aligned} G(s) &= 0 \\ &= \beta\xi \sinh i\gamma_n\left(\zeta \frac{i\gamma_n}{\beta} \cosh \frac{i\gamma_n}{\beta} + \sinh \frac{i\gamma_n}{\beta}\right) \\ &\quad + \cosh i\gamma_n\left(\zeta \frac{i\gamma_n}{\beta} \sinh \frac{i\gamma_n}{\beta} + \cosh \frac{i\gamma_n}{\beta}\right) \end{aligned}$$

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which becomes

$$\beta\xi \sin \gamma_n\left(\zeta \frac{\gamma_n}{\beta} \cos \frac{\gamma_n}{\beta} + \sin \frac{\gamma_n}{\beta}\right) + \cos \gamma_n\left(\zeta \frac{\gamma_n}{\beta} \sin \frac{\gamma_n}{\beta} - \cos \frac{\gamma_n}{\beta}\right) = 0 \tag{16}$$

Equation (16) is the characteristic equation for the problem. The inverse transforms of equations (11) then become

$$\begin{aligned} \frac{T_1 - T_o}{T_{aw} - T_o} &= 1 + \beta\xi \sum_{n=1}^{\infty} \frac{\left(\sin \gamma_n\right)\left(\sin \frac{\gamma_n}{\beta} \eta_1\right)}{K_n} e^{-\gamma_n^2\alpha_2} \\ &\quad - \sum_{n=1}^{\infty} \frac{\left(\cos \gamma_n\right)\left(\cos \frac{\gamma_n}{\beta} \eta_1\right)}{K_n} e^{-\gamma_n^2\alpha_2} \end{aligned} \tag{17a}$$

$$\frac{T_2 - T_o}{T_{aw} - T_o} = 1 - \sum_{n=1}^{\infty} \frac{\cos \gamma_n\eta_2}{K_n} e^{-\gamma_n\alpha_2} \tag{17b}$$

where

$$K_n = \frac{\gamma_n}{2}\left\{\cos \gamma_n\left[\xi\zeta\gamma_n \cos \frac{\gamma_n}{\beta} + \beta\xi \sin \frac{\gamma_n}{\beta} + \frac{\zeta}{\beta^2}\left(\beta \sin \frac{\gamma_n}{\beta} + \gamma_n \cos \frac{\gamma_n}{\beta}\right)\right.\right.$$

$$\left. + \frac{1}{\beta} \sin \frac{\gamma_n}{\beta}\right] + \sin \gamma_n\left[\cos \frac{\gamma_n}{\beta} - \frac{\zeta\gamma_n}{\beta} \sin \frac{\gamma_n}{\beta} + \frac{\xi\zeta}{\beta}\left(\beta \cos \frac{\gamma_n}{\beta}\right.\right.$$

$$\left.\left.\left. - \gamma_n \sin \frac{\gamma_n}{\beta}\right) + \xi \cos \frac{\gamma_n}{\beta}\right]\right\} \tag{18}$$

and the values of γ_n are the roots of equation (16).

<u>Variable adiabatic wall temperature</u>.- The exact solution to a heat-transfer problem with a step-function adiabatic wall temperature may be used to obtain solutions to problems where the adiabatic wall temperature varies with time. Duhamel's integral (ref. 4) gives the relationship between the response to a step-function input and an arbitrary input. The use of this relationship with the previous step-function solution (eqs. 17) gives the following equations in which T_{aw} is a function of time:

$$T_1 - T_o = \left(T_{aw} - T_o\right)_{\alpha_2=0}\left[1 + \beta\xi \sum_{n=1}^{\infty} \frac{\left(\sin \gamma_n\right)\left(\sin \frac{\gamma_n}{\beta} \eta_1\right)}{K_n} e^{-\gamma_n{}^2\alpha_2}\right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{\left(\cos \gamma_n\right)\left(\cos \frac{\gamma_n}{\beta} \eta_1\right)}{K_n} e^{-\gamma_n\alpha_2}\right]$$

$$+ \int_0^{\alpha_2}\left[1 + \beta\xi \sum_{n=1}^{\infty} \frac{\left(\sin \gamma_n\right)\left(\sin \frac{\gamma_n}{\beta} \eta_1\right)}{K_n} e^{-\gamma_n(\alpha_2-u)}\right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{\left(\cos \gamma_n\right)\left(\cos \frac{\gamma_n}{\beta} \eta_1\right)}{K_n} e^{-\gamma_n{}^2(\alpha_2-u)}\right] \frac{d\left(T_{aw} - T_o\right)}{du} du$$

(19a)

$$T_2 - T_o = \left(T_{aw} - T_o\right)_{\alpha_2=0}\left(1 - \sum_{n=1}^{\infty} \frac{\cos \gamma_n \eta_2}{K_n} e^{-\gamma_n{}^2\alpha_2}\right)$$

$$+ \int_0^{\alpha_2}\left[1 - \sum_{n=1}^{\infty} \frac{\cos \gamma_n \eta_2}{K_n} e^{-\gamma_n{}^2(\alpha_2 - u)}\right]\frac{d\left(T_{aw} - T_o\right)}{du}\, du \tag{19b}$$

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In these equations the adiabatic wall temperature, which is a function of time, is expressed as a function of the nondimensional time parameter α_2, and u is a dummy variable of integration representing time. It is understood that h remains constant. The characteristic equation for equations (19) is equation (16) and thus the roots γ_n are the same as those for constant T_{aw}.

Constant, prescribed surface temperature.- The preceding development has neglected the effects of radiation cooling. When the insulation surface temperature is high, a considerable amount of heat will be radiated to the surroundings instead of being conducted to the interior of the slab. The amount of heat radiated is dependent upon the fourth power of the absolute temperature of the surface, which results in a nonlinear heat-transfer problem. The resulting complexity of solution can be reduced by an approximation to the outer-surface boundary condition. (See ref. 2, appendix B.)

The heat transfer at the insulation outer surface is

$$q = q_r + q_c \tag{20}$$

where q is the amount of heat conducted to the surface through the boundary layer, q_r is the heat radiated away from the surface to the surroundings, and q_c is the heat conducted into the structure. Equation (20) may be written as

$$h\left(T_{aw} - T_1\right) = \sigma F\left(T_1{}^4 - T_{fs}{}^4\right) + k_1 \left.\frac{\partial T_1}{\partial \eta_1}\right|_{\eta_1=1} \tag{21}$$

where T_1 is the temperature of the outside surface of the insulation,

T_{fs} is the effective temperature of free space, σ is the Stefan-Boltzmann constant, and F is the geometric radiation factor which includes emissivity. The equation is useful when $T_{aw} > 3,000^{\circ}$ R, and $T_1 > 2,000^{\circ}$ R. For insulating materials $k_1 \left.\frac{\partial T_1}{\partial \eta_1}\right|_{\eta_1=1}$ will be small with respect to the radiation term and will be neglected in equation (21). Also,

$$T_{fs}^4 \ll T_1^4$$

and T_{fs} will be ignored. There remains

$$h\left(T_{aw} - T_1\right) = \sigma F T_1^4$$

or

$$T_1\left(1 + \frac{\sigma F}{h} T_1^3\right) = T_{aw}$$

which may be written as

$$T_{eq}\left(1 + \frac{\sigma F}{h} T_{eq}^3\right) = T_{aw} \tag{22}$$

where T_{eq} is the equilibrium temperature of the outside surface of the insulation. The resulting heat-transfer problem is one in which the temperature of the outside surface is prescribed as T_{eq}.

The solution of the problem with the prescribed surface temperature may be obtained by manipulation of equations (17). In equations (17) T_{aw} is prescribed. The aerodynamic heat-transfer coefficient h requires that $T_1\left(1,\alpha_2\right) < T_{aw}$ unless $h \to \infty$. If $h \to \infty$, there is no gradient between T_{aw} and $T_1\left(1,\alpha_2\right)$, and $T_1\left(1,\alpha_2\right) = T_{aw}$. Equations (16) and (17) with $\lim\limits_{h\to\infty} = \lim\limits_{\zeta\to 0}$ are then the solutions to a heat-transfer problem with prescribed surface temperature. It is necessary now only to change T_{aw} to T_{eq} in equations (17) to complete the manipulation. The result for the metal plate only is

$$\frac{T_2 - T_o}{T_{eq} - T_o} = 1 - \sum_{\eta=1}^{\infty} \frac{\cos \gamma_n \eta_2}{H_n} e^{-\gamma_n^2 \alpha_2} \tag{23}$$

where

$$H_n = \frac{\gamma_n}{2}\left[\left(\beta\xi + \frac{1}{\beta}\right) \cos \gamma_n \sin \frac{\gamma_n}{\beta} + (1 + \xi)\sin \gamma_n \cos \frac{\gamma_n}{\beta}\right] \tag{24}$$

and where the roots γ_n are determined from

$$\beta\xi \tan \gamma_n \tan \frac{\gamma_n}{\beta} = 1 \tag{25}$$

which is obtained from equation (16) by taking the limit as $\zeta \rightarrow 0$.

Variable prescribed surface temperature.- In a manner similar to the solution for the problem of variable T_{aw}, the solution to the problem involving a variable T_{eq} may be found. Applying Duhamel's integral to equation (23) results in

$$T_2 - T_o = \left(T_{eq} - T_o\right)_{\alpha_2=0}\left(1 - \sum_{n=1}^{\infty} \frac{\cos \gamma_n\eta_2}{H_n} e^{-\gamma_n{}^2\alpha_2}\right)$$

$$+ \int_0^{\alpha_2}\left[1 - \sum_{n=1}^{\infty} \frac{\cos \gamma_n\eta_2}{H_n} e^{-\gamma_n{}^2(\alpha_2-u)}\right] \frac{d\left(T_{eq} - T_o\right)}{du} du \tag{26}$$

where γ_n is determined from equation (25).

In many instances the gradient through the metal plate is small, and only the average temperature of the plate is of interest. The average temperature of the metal may be found by integrating equation (17b) over the thickness and then dividing by the thickness. There results

$$\frac{\bar{T}_2 - T_o}{T_{aw} - T_o} = 1 - \sum_{n=1}^{\infty} \frac{\cos \gamma_n}{\gamma_n K_n} e^{-\gamma_n{}^2\alpha_2} \quad \left(T_{aw} = \text{Constant}\right) \tag{27}$$

where $\bar{T}_2$ is the average temperature of the plate and where the values of γ_n are the roots of equation (16). The equilibrium temperature equation is

$$\frac{\overline{T}_2 - T_o}{T_{eq} - T_o} = 1 - \sum_{n=1}^{\infty} \frac{\sin \gamma_n}{\gamma_n H_n} e^{-\gamma_n{}^2 \alpha_2} \qquad \left(T_{eq} = \text{Constant}\right) \tag{28}$$

where the roots γ_n are determined from equation (25). The solution to the cases with variable T_{aw} is:

$$\overline{T}_2 - T_o = \left(T_{aw} - T_o\right)_{\alpha_2=0}\left(1 - \sum_{n=1}^{\infty} \frac{\sin \gamma_n}{\gamma_n K_n} e^{-\gamma_n{}^2 \alpha_2}\right)$$

$$+ \int_0^{\alpha_2}\left[1 - \sum_{n=1}^{\infty} \frac{\sin \gamma_n}{\gamma_n K_n} e^{-\gamma_n{}^2 (\alpha_2 - u)}\right] \frac{d\left(T_{aw} - T_o\right)}{du}\, du \tag{29}$$

where $T_{aw} = f(\alpha_2)$ and values of γ_n are from equation (16). Similarly, for variable T_{eq},

$$\overline{T}_2 - T_o = \left(T_{eq} - T_o\right)_{\alpha_2=0}\left(1 - \sum_{n=1}^{\infty} \frac{\sin \gamma_n}{\gamma_n H_n} e^{-\gamma_n{}^2 \alpha_2}\right)$$

$$+ \int_0^{\alpha_2}\left[1 - \sum_{n=1}^{\infty} \frac{\sin \gamma_n}{\gamma_n H_n} e^{-\gamma_n{}^2 (\alpha_2 - u)}\right] \frac{d\left(T_{eq} - T_o\right)}{du}\, du \tag{30}$$

where $T_{eq} = g(\alpha_2)$ and values of γ_n are from equation (25).

Approximate Equations

First approximate solution.- In order to obtain the first approximate solution to the exact solutions in the preceding section, the following assumptions are made:

(1) The insulation has no thermal capacity

(2) There is no temperature gradient through the metal plate

For the case of constant adiabatic wall temperature, the governing equations may be obtained by considering heat balances for the system sketched in figure 1(b).

$$h\left(T_{aw} - T_1\right) = \frac{k_1}{t_1}\left(T_1 - T_2\right) \tag{31a}$$

$$\frac{k_1}{t_1}\left(T_1 - T_2\right) = c_2 w_2 t_2 \frac{dT_2}{d\tau} \tag{31b}$$

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In nondimensional form, these equations are

$$T_{aw} - T_1 = \zeta\left(T_1 - T_2\right) \tag{32a}$$

$$T_1 - T_2 = \frac{dT_2}{d\lambda} \tag{32b}$$

where

$$\zeta = \frac{k_1}{ht_1}$$

and

$$\lambda = \frac{k_1 \tau}{c_2 w_2 t_2 t_1}$$

The solution for the metal temperature is

$$\begin{aligned}\frac{T_2 - T_o}{T_{aw} - T_o} &= 1 - \exp\left(-\frac{\lambda}{1 + \zeta}\right) \\ &= 1 - \exp\left[-\frac{k_1 \tau}{c_2 w_2 t_2 t_1\left(1 + \frac{k_1}{ht_1}\right)}\right]\end{aligned} \tag{33}$$

Equation (33) can also be obtained by restricting equation (17b) to the first series term, allowing c_1 to approach zero in equation (16), and then assuming γ_n small so that $\gamma_n \tan \gamma_n \approx \gamma_n^2$.

The solution for the case of variable adiabatic wall temperature is

$$T_2 - T_o = \left(T_{aw} - T_o\right)_{\lambda=0}\left[1 - \exp\left(- \frac{\lambda}{1 + \zeta}\right)\right]$$

$$+ \int_0^{\lambda}\left[1 - \exp\left(- \frac{\lambda - u}{1 + \zeta}\right)\right]\frac{d\left(T_{aw} - T_o\right)}{du}\,du \tag{34}$$

where λ is used as the time variable and u is the dummy variable of integration.

Second approximate solution.- The second approximate solution admits that the thermal capacity of the insulation reduces the temperature rise of the metal structure. The assumptions are that:

(1) There is no thermal gradient through the metal plate

(2) The thermal gradient through the insulation is linear

The second assumption is exactly true only in the steady-state condition where the metal temperature is constant and where transient conditions due to a change in adiabatic wall temperature have died out. The temperature of a well insulated plate changes slowly, so that the approximation should be satisfactory if T_{aw} changes slowly with time. The governing equations are:

$$\frac{1}{2} c_1 w_1 t_1 \frac{dT_1}{d\tau} = h\left(T_{aw} - T_1\right) - \frac{k_1}{t_1}\left(T_1 - T_2\right) \tag{35a}$$

$$\left(\frac{1}{2} c_1 w_1 t_1 + c_2 w_2 t_2\right)\frac{dT_2}{d\tau} = \frac{k_1}{t_1}\left(T_1 - T_2\right) \tag{35b}$$

The solutions to equations (35) are

$$T_1 = A_{11}e^{-\phi_1 \tau} + A_{12}e^{-\phi_2 \tau} + T_{aw} \tag{36a}$$

$$T_2 = A_{21}e^{-\phi_1 \tau} + A_{22}e^{-\phi_2 \tau} + T_{aw} \tag{36b}$$

where

$$\phi_j = \left[\frac{k_1}{c_1w_1t_1{}^2} + \frac{h}{c_1w_1t_1} + \frac{k_1}{t_1}\left(\frac{1}{c_1w_1t_1 + 2c_2w_2t_2}\right)\right]$$

$$- (-1)^j \left\{\left[\frac{k_1}{c_1w_1t_1{}^2} + \frac{h}{c_1w_1t_1} + \frac{k_1}{t_1}\left(\frac{1}{c_1w_1t_1 + 2c_2w_2t_2}\right)\right]^2\right.$$

$$\left. - \frac{4hk_1}{t_1\left(c_1w_1t_1 + 2c_2w_2t_2\right)}\right\}^{1/2} \tag{37a}$$

$$A_{21} = -\frac{\phi_2}{\phi_1 - \phi_2}\left(T_o - T_{aw}\right) \tag{37b}$$

$$A_{22} = \frac{\phi_1}{\phi_1 - \phi_2}\left(T_o - T_{aw}\right) \tag{37c}$$

The term in equations (36) with the exponent containing ϕ_1 is small compared with the other term except when T_{aw} is transient. For flight times long compared with the initial transient period, the ϕ_1 term can be neglected.

Expanding the square-root term in equation (37a) to two terms yields

$$\left\{\left[\frac{k_1}{c_1w_1t_1{}^2} + \frac{h}{c_1w_1t_1} + \frac{k_1}{t_1}\left(\frac{1}{c_1w_1t_1 + 2c_2w_2t_2}\right)\right]^2 - \frac{4hk_1}{t_1\left(c_1w_1t_1 + 2c_2w_2t_2\right)}\right\}^{1/2}$$

$$= \frac{k_1/t_1}{c_2w_2t_2\left(1 + \dfrac{k_1}{ht_1}\right)}\left[\frac{1}{1 + \left(\dfrac{1 + \dfrac{1}{2}\dfrac{ht_1}{k_1}}{1 + \dfrac{ht_1}{k_1}}\right)\left(\dfrac{c_1w_1t_1}{c_2w_2t_2}\right)}\right] \tag{38}$$

In many practical cases, the term k_1/ht_1 is small compared to 1.0. With this assumption, a second approximate solution to equation (17b) might be

$$\frac{T_2 - T_o}{T_{aw} - T_o} = 1 - \exp\left[-\frac{k_1\tau}{c_2w_2t_2t_1\left(1 + \frac{k_1}{ht_1}\right)\left(1 + \frac{c_1w_1t_1}{2c_2w_2t_2}\right)}\right] \tag{39}$$

This equation is the same as equation (33) except for the term $1 + \dfrac{c_1w_1t_1}{2c_2w_2t_2}$, which takes into account the effect of the thermal capacity of the insulation.

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The solution when T_{aw} is variable is

$$T_2 - T_o = \left(T_{aw} - T_o\right)_{\tau=0}\left\{1 - \exp\left[-\frac{k_1\tau}{c_2w_2t_2t_1\left(1 + \frac{k_1}{ht_1}\right)\left(1 + \frac{c_1w_1t_1}{2c_2w_2t_2}\right)}\right]\right\}$$

$$+ \int_0^\tau\left\{1 - \exp\left[-\frac{k_1(\tau - u)}{c_2w_2t_2t_1\left(1 + \frac{k_1}{ht_1}\right)\left(1 + \frac{c_1w_1t_1}{2c_2w_2t_2}\right)}\right]\right\}\frac{d\left(T_{aw} - T_o\right)}{du}\,du \tag{40}$$

where u is a dummy variable of integration.

RESULTS AND DISCUSSION

The first nine roots of equation (16) are given in table I for various values of the nondimensional parameters $\rho = \dfrac{c_2w_2t_2}{c_1w_1t_1}$, $\xi = \dfrac{k_2t_1}{k_1t_2}$, and $\zeta = \dfrac{k_1}{ht_1}$, where k is thermal conductivity, c is specific heat, w is weight density, t is thickness, h is the heat-transfer coefficient, and the subscripts 1 and 2 refer to the insulation slab and the metal slab, respectively. Table II gives the roots of the characteristic equation for the special case of an infinite heat-transfer coefficient (eq. (25), $\zeta = 0$), that is, solutions to a heat-transfer problem with prescribed surface temperature. Interpolation between various values of the roots is permitted for intermediate untabulated values of the parameters. This interpolation is not necessarily linear; one method of interpolation would be to construct a carpet plot of the desired roots around the vicinity under consideration.

Figure 2 is a carpet plot of solutions to equation (27), the average temperature of the metal slab for a step-function adiabatic wall temperature. In this and subsequent figures, note that the ordinates have been modified to represent only the summation terms in the temperature equations. (A method of interpolation on carpet plots is demonstrated in fig. 8 of ref. 5.) Figure 3 presents solutions to equation (17b) for $\eta_2 = 0$; this is the temperature at the back surface of the metal slab. Figure 4 presents solutions to equation (17b) for the temperatures at the interface of insulation and slab $\left(\eta_2 = 1, \ \eta_1 = 0\right)$, and figure 5 is a carpet plot of equation (17a) for the insulation outside surface temperature $\left(\eta_1 = 1\right)$. Figure 6 shows the average temperature of the back slab for the case of constant prescribed surface temperature of the insulation $(h \to \infty, \ \zeta \to 0)$.

In order to compress the time scale to a convenient size the solutions were plotted with a nondimensional time scale of

$$\omega = \frac{k_1 \tau}{c_2 w_2 t_2 t_1 \left(1 + \frac{k_1}{h t_1}\right)\left(1 + \frac{c_1 w_1 t_1}{2 c_2 w_2 t_2}\right)}$$

$$= \frac{\lambda}{(1 + \zeta)\left(1 + \frac{1}{2\rho}\right)} \tag{41}$$

which will be recognized as the nondimensional time parameter from the second approximate solution. (See eq. (39).)

Because of the extremely small temperature difference between the interface and the exposed surface of the metal when $\xi \geqq 100$, only one plot is required in figures 2 to 4 to show T_2 at $\eta_2 = 1$ and $\eta_2 = 0$. Also, because T_1 approaches T_{aw} very rapidly when $\zeta \leqq 0.1$, values of T_1 are not included in figure 5. The case $\eta_2 = 0$, $\zeta = 1.0$, $\rho = 100$ has been omitted from figure 3(a) because the scales of the carpet are such that, for this case, the carpet "folds back" on itself near $\zeta = 1.0$.

Because the nondimensional time parameter used in constructing the charts is that of the second approximate solution, the accuracy of the second approximate solution may be rapidly compared with any of the curves in figure 2 by comparing the curve with

$$\frac{T_{aw} - T_2}{T_{aw} - T_o} = e^{-\omega} \tag{42a}$$

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or

$$\log_e\left(\frac{T_{aw} - T_2}{T_{aw} - T_o}\right) = -\omega \tag{42b}$$

which is a straight line on the figure. The dashed lines shown in figure 2 are plots of equation (42b). It can be seen from equation (42b) that the slope of the second approximate solution in this plot will be a constant, independent of ρ, ξ, and ζ.

A comparison of solutions for the case of $\zeta = 0$ (fig. 6) shows that if $\frac{c_2 w_2 t_2}{c_1 w_1 t_1} \approx 1$ the error in $\overline{T}_2$ at the beginning of the heating period is about $0.05(T_{eq} - T_o)$. Because the approximate solutions neglect the thermal capacity of the insulation, these solutions give a higher temperature rise than the exact solution. As the heating period progresses the second approximate solution reaches the exact solution at the time when $T_2 = T_{eq} - 0.37(T_{eq} - T_o)$. When $T_2 = T_{eq} - 0.1(T_{eq} - T_o)$ the second approximate solution is $0.015(T_{eq} - T_o)$ lower than the exact solution.

When $\rho = \frac{c_2 w_2 t_2}{c_1 w_1 t_1} \approx 10$, the second approximate solution and the exact solution agree within $0.005(T_{eq} - T_o)$ for times when $T_2 \leq T_{eq} - 0.05(T_{eq} - T_o)$. This solution improves as $\rho = \frac{c_2 w_2 t_2}{c_1 w_1 t_1} \to \infty$.

As a result of the choice of the time parameter, the slope of the first approximate solution, as it is plotted in the figures, changes with ρ. As a function of ω, the first approximate solution can be written as

$$\frac{T_{aw} - T_2}{T_{aw} - T_o} = e^{-\frac{\lambda}{1+\zeta}} = e^{-\omega\left(1+\frac{1}{2\rho}\right)}$$

The first approximate solution agrees with the exact solution for the average metal temperature within 5 percent when the following criteria are satisfied:

$$\rho \geqq 10.0$$

$$\xi \geqq 100.0$$

$$\zeta \leqq 0.1$$

$$\omega \leqq 1.5$$

The accuracy of this solution increases with an increase in ρ or a decrease in ω. For $\rho = 10$, the error at $\omega = 4.0$ is approximately 15 percent. However, for $\rho = 100$, the error up to $\omega = 4.0$ is negligible.

The second approximate solution is within 5 percent of the exact solution when

$$\rho \geqq 1.0$$

$$\xi \geqq 100.0$$

$$\zeta \leqq 0.1$$

$$\omega \leqq 1.5$$

When $\omega = 4.0$ and $\rho = 1.0$, the error is approximately 35 percent. However, when $\rho = 100$, the error is negligible up to $\omega = 4.0$ for all values of ζ, and when $\rho = 10$, the error is only 3 percent at $\omega = 4.0$, with ζ less than 0.1. Although errors of 15 and 35 percent seem large, when $\omega = 4.0$ $\left(T_{aw} - T_2\right)$ is approaching zero, so that the error in degrees is relatively small.

As long as ρ and ξ are sufficiently large, the first approximate solution is adequate and will be useful in many cases because of its simplicity. However, for values of ρ less than 10, the second approximate solution will be more accurate.

CONCLUDING REMARKS

Two approximate solutions for the temperatures in an insulated structure have been compared with the exact solution. The approximations are shown to be sufficiently accurate for use in many practical problems. These approximations are good as long as the ratio of thermal capacity of the metal to that of the insulation and the ratio of the conductance of the metal to that of the insulation are large compared to 1.0 and the ratio of the conductance of the insulation to the heat-transfer coefficient is small compared to 1.0.

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Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., July 21, 1960.

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TABLE I.- ROOTS, γ_n, OF CHARACTERISTIC EQUATION (16)

(a) $\xi = 1.0$

ρ	n	γ_n for ζ = : 0.0001	0.001	0.01	0.1	1.0	10.0	100.0	1,000.0
0.01	1	0.1555×10^{0}	0.1554×10^{0}	0.1540×10^{0}	0.1416×10^{0}	0.8549×10^{-1}	0.3095×10^{-1}	0.9934×10^{-2}	0.3146×10^{-2}
	2	.4662	.4658	.4617	.4264	$.3393 \times 10^{0}$	$.3141 \times 10^{0}$	$.3113 \times 10^{0}$	$.3110 \times 10^{0}$
	3	.7755	.7749	.7681	.7147	.6365	.6228	.6213	.6212
	4	$.1081 \times 10$	$.1080 \times 10$	$.1071 \times 10$	$.1005 \times 10$	.9395	.9302	.9293	.9292
	5	.1368	.1367	.1357	.1289	$.1237 \times 10$	$.1230 \times 10$	$.1229 \times 10$	$.1229 \times 10$
	6	.1571	.1570	.1563	.1519	.1489	.1485	.1485	.1485
	7	.1773	.1772	.1759	.1693	.1661	.1657	.1657	.1657
	8	.2060	.2059	.2041	.1956	.1917	.1913	.1912	.1912
	9	.2366	.2364	.2343	.2253	.2217	.2213	.2212	.2212
0.1	1	0.4489×10^{0}	0.4485×10^{0}	0.4450×10^{0}	0.4123×10^{0}	0.2557×10^{0}	0.9362×10^{-1}	0.3010×10^{-1}	0.9533×10^{-2}
	2	$.1250 \times 10$	$.1249 \times 10$	$.1242 \times 10$	$.1174 \times 10$	.9599	$.8876 \times 10^{0}$	$.8792 \times 10^{0}$	$.8784 \times 10^{0}$
	3	.1792	.1791	.1781	.1704	$.1562 \times 10$	$.1534 \times 10$	$.1531 \times 10$	$.1530 \times 10$
	4	.2550	.2548	.2528	.2374	.2170	.2137	.2134	.2134
	5	.3446	.3443	.3415	.3218	.3025	.2998	.2995	.2995
	6	.4280	.4277	.4250	.4059	.3906	.3887	.3885	.3885
	7	.4843	.4840	.4818	.4685	.4600	.4590	.4589	.4589
	8	.5551	.5547	.5505	.5283	.5168	.5155	.5154	.5154
	9	.6441	.6436	.6384	.6127	.6006	.5992	.5991	.5991
1.0	1	0.7854×10^{0}	0.7850×10^{0}	0.7815×10^{0}	0.7480×10^{0}	0.5384×10^{0}	0.2164×10^{0}	0.7048×10^{-1}	0.2235×10^{-1}
	2	$.2356 \times 10$	$.2355 \times 10$	$.2344 \times 10$	$.2246 \times 10$	$.1822 \times 10$	$.1602 \times 10$	$.1574 \times 10$	$.1571 \times 10$
	3	.3927	.3925	.3907	.3748	.3289	.3157	.3143	.3142
	4	.5498	.5495	.5470	.5256	.4815	.4723	.4713	.4712
	5	.7068	.7065	.7033	.6771	.6361	.6291	.6284	.6283
	6	.8639	.8635	.8597	.8293	.7917	.7860	.7855	.7854
	7	$.1021 \times 10^{2}$	$.1021 \times 10^{2}$	$.1016 \times 10^{2}$	.9822	.9477	.9430	.9425	.9425
	8	.1178	.1178	.1172	$.1136 \times 10^{2}$	$.1104 \times 10^{2}$	$.1100 \times 10^{2}$	$.1100 \times 10^{2}$	$.1100 \times 10^{2}$
	9	.1335	.1335	.1329	.1290	.1261	.1257	.1257	.1257
10.0	1	0.8526×10^{0}	0.8523×10^{0}	0.8494×10^{0}	0.8219×10^{0}	0.6397×10^{0}	0.2850×10^{0}	0.9479×10^{-1}	0.3013×10^{-1}
	2	$.3321 \times 10$	$.3321 \times 10$	$.3317 \times 10$	$.3285 \times 10$	$.3085 \times 10$	$.2835 \times 10$	$.2784 \times 10$	$.2778 \times 10$
	3	.6159	.6158	.6152	.6093	.5546	.4939	.4850	.4841
	4	.8748	.8746	.8721	.8446	.7137	.6781	.6751	.6748
	5	$.1049 \times 10^{2}$	$.1048 \times 10^{2}$	$.1044 \times 10^{2}$	$.1011 \times 10^{2}$	.9570	.9481	.9472	.9472
	6	.1281	.1281	.1279	.1267	$.1239 \times 10^{2}$	$.1230 \times 10^{2}$	$.1228 \times 10^{2}$	$.1228 \times 10^{2}$
	7	.1563	.1563	.1562	.1548	.1480	.1454	.1451	.1451
	8	.1832	.1831	.1827	.1775	.1649	.1631	.1630	.1630
	9	.2015	.2014	.2005	.1946	.1900	.1895	.1895	.1894
100.0	1	0.8595×10^{0}	0.8592×10^{0}	0.8564×10^{0}	0.8297×10^{0}	0.6519×10^{0}	0.2958×10^{0}	0.9886×10^{-1}	0.3145×10^{-1}
	2	$.3416 \times 10$	$.3416 \times 10$	$.3413 \times 10$	$.3392 \times 10$	$.3274 \times 10$	$.3141 \times 10$	$.3113 \times 10$	$.3110 \times 10$
	3	.6416	.6416	.6415	.6400	.6323	.6233	.6214	.6212
	4	.9496	.9496	.9495	.9483	.9412	.9317	.9295	.9292
	5	$.1260 \times 10^{2}$	$.1260 \times 10^{2}$	$.1260 \times 10^{2}$	$.1259 \times 10^{2}$	$.1250 \times 10^{2}$	$.1234 \times 10^{2}$	$.1230 \times 10^{2}$	$.1229 \times 10^{2}$
	6	.1571	.1571	.1571	.1569	.1556	.1506	.1487	.1485
	7	.1882	.1882	.1882	.1880	.1844	.1681	.1659	.1657
	8	.2192	.2192	.2192	.2188	.2035	.1917	.1913	.1912
	9	.2500	.2500	.2500	.2490	.2237	.2214	.2213	.2212

TABLE I.- ROOTS, γ_n, OF CHARACTERISTIC EQUATION (16) - Continued

(b) $\xi = 10.0$

ρ	n	γ_n for ζ = :							
		0.0001	0.001	0.01	0.1	1.0	10.0	100.0	1,000.0
0.01	1	0.4918×10^{-1}	0.4913×10^{-1}	0.4870×10^{-1}	0.4478×10^{-1}	0.2703×10^{-1}	0.9786×10^{-2}	0.3141×10^{-2}	0.9949×10^{-3}
	2	$.1475 \times 10^{0}$	$.1474 \times 10^{0}$	$.1461 \times 10^{0}$	$.1349 \times 10^{0}$	$.1073 \times 10^{0}$	$.9936 \times 10^{-1}$	$.9846 \times 10^{-1}$	$.9837 \times 10^{-1}$
	3	.2458	.2456	.2435	.2264	.2016	$.1972 \times 10^{0}$	$.1968 \times 10^{0}$	$.1967 \times 10^{0}$
	4	.3441	.3438	.3408	.3194	.2983	.2953	.2950	.2950
	5	.4423	.4419	.4381	.4137	.3958	.3935	.3933	.3933
	6	.5404	.5399	.5352	.5088	.4934	.4916	.4914	.4914
	7	.6384	.6378	.6323	.6045	.5912	.5897	.5895	.5895
	8	.7362	.7355	.7292	.7006	.6888	.6875	.6874	.6874
	9	.8338	.8330	.8259	.7968	.7863	.7852	.7851	.7851
0.1	1	0.1428×10^{0}	0.1427×10^{0}	0.1415×10^{0}	0.1310×10^{0}	0.8094×10^{-1}	0.2961×10^{-1}	0.9517×10^{-2}	0.3015×10^{-2}
	2	.4284	.4280	.4245	.3943	$.3137 \times 10^{0}$	$.2887 \times 10^{0}$	$.2859 \times 10^{0}$	$.2856 \times 10^{0}$
	3	.7139	.7134	.7076	.6609	.5866	.5728	.5714	.5712
	4	.9995	.9987	.9906	.9314	.8672	.8579	.8569	.8568
	5	$.1285 \times 10$	$.1284 \times 10$	$.1274 \times 10$	$.1205 \times 10$	$.1150 \times 10$	$.1143 \times 10$	$.1142 \times 10$	$.1142 \times 10$
	6	.1571	.1569	.1557	.1482	.1434	.1429	.1428	.1428
	7	.1856	.1855	.1840	.1761	.1719	.1714	.1714	.1714
	8	.2142	.2140	.2123	.2041	.2004	.2000	.1999	.1999
	9	.2427	.2425	.2406	.2322	.2289	.2285	.2285	.2285
1.0	1	0.2696×10^{0}	0.2694×10^{0}	0.2680×10^{0}	0.2540×10^{0}	0.1753×10^{0}	0.6869×10^{-1}	0.2229×10^{-1}	0.7069×10^{-2}
	2	$.1050 \times 10$	$.1049 \times 10$	$.1041 \times 10$	.9674	.7325	$.6395 \times 10^{0}$	$.6282 \times 10^{0}$	$.6270 \times 10^{0}$
	3	.1947	.1946	.1930	$.1800 \times 10$	$.1555 \times 10$	$.1504 \times 10$	$.1498 \times 10$	$.1498 \times 10$
	4	.2766	.2765	.2747	.2607	.2416	.2386	.2382	.2382
	5	.3316	.3314	.3298	.3185	.3077	.3062	.3060	.3060
	6	.4050	.4047	.4016	.3815	.3663	.3645	.3643	.3643
	7	.4943	.4939	.4900	.4664	.4513	.4495	.4493	.4493
	8	.5792	.5788	.5749	.5526	.5402	.5387	.5386	.5386
	9	.6371	.6368	.6339	.6196	.6125	.6117	.6116	.6116
10.0	1	0.3063×10^{0}	0.3061×10^{0}	0.3048×10^{0}	0.2920×10^{0}	0.2157×10^{0}	0.9115×10^{-1}	0.3001×10^{-1}	0.9530×10^{-2}
	2	$.2835 \times 10$	$.2834 \times 10$	$.2821 \times 10$	$.2681 \times 10$	$.1991 \times 10$	$.1627 \times 10$	$.1577 \times 10$	$.1571 \times 10$
	3	.3448	.3446	.3431	.3322	.3170	.3144	.3142	.3142
	4	.5977	.5974	.5946	.5638	.4895	.4732	.4714	.4713
	5	.6589	.6586	.6558	.6404	.6298	.6285	.6283	.6283
	6	.9118	.9114	.9070	.8618	.7967	.7866	.7855	.7854
	7	.9731	.9726	.9686	.9513	.9434	.9426	.9425	.9425
	8	$.1226 \times 10^{2}$	$.1225 \times 10^{2}$	$.1219 \times 10^{2}$	$.1163 \times 10^{2}$	$.1108 \times 10^{2}$	$.1100 \times 10^{2}$	$.1100 \times 10^{2}$	$.1100 \times 10^{2}$
	9	.1287	.1287	.1281	.1263	.1257	.1257	.1257	.1257
100.0	1	0.3106×10^{0}	0.3104×10^{0}	0.3091×10^{0}	0.2965×10^{0}	0.2211×10^{0}	0.9477×10^{-1}	0.3131×10^{-1}	0.9945×10^{-2}
	2	$.3162 \times 10$	$.3162 \times 10$	$.3161 \times 10$	$.3158 \times 10$	$.3135 \times 10$	$.3103 \times 10$	$.3096 \times 10$	$.3095 \times 10$
	3	.6269	.6269	.6269	.6261	.6057	.5119	.4962	.4946
	4	.9276	.9274	.9257	.8896	.6640	.6360	.6352	.6351
	5	$.1007 \times 10^{2}$	$.1006 \times 10^{2}$	.9992	.9591	.9441	.9431	.9430	.9430
	6	.1259	.1259	$.1259 \times 10^{2}$	$.1258 \times 10^{2}$	$.1255 \times 10^{2}$	$.1253 \times 10^{2}$	$.1253 \times 10^{2}$	$.1253 \times 10^{2}$
	7	.1570	.1570	.1570	.1568	.1531	.1488	.1483	.1482
	8	.1876	.1876	.1875	.1811	.1594	.1582	.1581	.1581
	9	.1992	.1991	.1975	.1897	.1887	.1886	.1886	.1886

TABLE I.- ROOTS, γ_n, OF CHARACTERISTIC EQUATION (16) - Continued

(c) $\xi = 100$

ρ	n	γ_n for ζ = :							
		0.0001	0.001	0.01	0.1	1.0	10.0	100.0	1,000.0
0.01	1	0.1555×10^{-1}	0.1554×10^{-1}	0.1540×10^{-1}	0.1416×10^{-1}	0.8549×10^{-2}	0.3095×10^{-2}	0.9934×10^{-3}	0.3146×10^{-3}
	2	.4665	.4661	.4620	.4266	$.3394 \times 10^{-1}$	$.3142 \times 10^{-1}$	$.3114 \times 10^{-1}$	$.3111 \times 10^{-1}$
	3	.7775	.7769	.7700	.7161	.6375	.6237	.6223	.6221
	4	$.1089 \times 10^{0}$	$.1088 \times 10^{0}$	$.1078 \times 10^{0}$	$.1010 \times 10^{0}$	.9436	.9342	.9333	.9332
	5	.1400	.1398	.1386	.1309	$.1252 \times 10^{0}$	$.1245 \times 10^{0}$	$.1244 \times 10^{0}$	$.1244 \times 10^{0}$
	6	.1711	.1709	.1694	.1610	.1562	.1556	.1555	.1555
	7	.2022	.2020	.2002	.1914	.1872	.1867	.1866	.1866
	8	.2333	.2331	.2310	.2219	.2182	.2178	.2177	.2177
	9	.2644	.2641	.2619	.2526	.2492	.2489	.2488	.2488
0.1	1	0.4518×10^{-1}	0.4514×10^{-1}	0.4477×10^{-1}	0.4143×10^{-1}	0.2560×10^{-1}	0.9363×10^{-2}	0.3010×10^{-2}	0.9533×10^{-3}
	2	$.1361 \times 10^{0}$	$.1360 \times 10^{0}$	$.1349 \times 10^{0}$	$.1251 \times 10^{0}$	.9942	$.9151 \times 10^{-1}$	$.9061 \times 10^{-1}$	$.9052 \times 10^{-1}$
	3	.2283	.2281	.2262	.2109	$.1869 \times 10^{0}$	$.1825 \times 10^{0}$	$.1821 \times 10^{0}$	$.1820 \times 10^{0}$
	4	.3220	.3217	.3190	.2993	.2784	.2753	.2750	.2750
	5	.4170	.4166	.4130	.3900	.3719	.3696	.3694	.3694
	6	.5129	.5124	.5080	.4826	.4669	.4651	.4649	.4648
	7	.6095	.6089	.6037	.5766	.5629	.5613	.5612	.5611
	8	.7066	.7060	.6999	.6716	.6595	.6582	.6580	.6580
	9	.8040	.8033	.7965	.7675	.7566	.7555	.7554	.7553
1.0	1	0.8595×10^{-1}	0.8590×10^{-1}	0.8542×10^{-1}	0.8089×10^{-1}	0.5558×10^{-1}	0.2173×10^{-1}	0.7050×10^{-2}	0.2235×10^{-2}
	2	$.3415 \times 10^{0}$	$.3413 \times 10^{0}$	$.3385 \times 10^{0}$	$.3137 \times 10^{0}$	$.2364 \times 10^{0}$	$.2065 \times 10^{0}$	$.2028 \times 10^{0}$	$.2025 \times 10^{0}$
	3	.6416	.6410	.6355	.5899	.5084	.4917	.4900	.4898
	4	.9495	.9487	.9404	.8787	.8073	.7964	.7953	.7952
	5	$.1260 \times 10$	$.1259 \times 10$	$.1248 \times 10$	$.1174 \times 10$	$.1113 \times 10$	$.1105 \times 10$	$.1105 \times 10$	$.1105 \times 10$
	6	.1571	.1569	.1556	.1474	.1422	.1416	.1415	.1415
	7	.1882	.1880	.1864	.1777	.1732	.1727	.1726	.1726
	8	.2192	.2190	.2171	.2081	.2042	.2038	.2037	.2037
	9	.2500	.2498	.2476	.2385	.2351	.2347	.2346	.2346
10.0	1	0.9821×10^{-1}	0.9816×10^{-1}	0.9771×10^{-1}	0.9350×10^{-1}	0.6867×10^{-1}	0.2886×10^{-1}	0.9491×10^{-2}	0.3014×10^{-2}
	2	$.9998 \times 10^{0}$	$.9989 \times 10^{0}$	$.9901 \times 10^{0}$	$.9124 \times 10^{0}$	$.6525 \times 10^{0}$	$.5324 \times 10^{0}$	$.5163 \times 10^{0}$	$.5146 \times 10^{0}$
	3	$.1982 \times 10$	$.1981 \times 10$	$.1963 \times 10$	$.1819 \times 10$	$.1554 \times 10$	$.1498 \times 10$	$.1492 \times 10$	$.1491 \times 10$
	4	.2933	.2931	.2909	.2732	.2510	.2475	.2471	.2471
	5	.3184	.3183	.3177	.3150	.3129	.3126	.3126	.3126
	6	.3982	.3979	.3944	.3716	.3530	.3506	.3504	.3504
	7	.4964	.4960	.4916	.4660	.4495	.4475	.4473	.4473
	8	.5933	.5928	.5879	.5614	.5473	.5457	.5455	.5455
	9	.6300	.6300	.6294	.6266	.6247	.6244	.6244	.6244
100.0	1	0.9966×10^{-1}	0.9962×10^{-1}	0.9917×10^{-1}	0.9503×10^{-1}	0.7045×10^{-1}	0.3001×10^{-1}	0.9901×10^{-2}	0.3145×10^{-2}
	2	$.3042 \times 10$	$.3040 \times 10$	$.3026 \times 10$	$.2834 \times 10$	$.2025 \times 10$	$.1631 \times 10$	$.1577 \times 10$	$.1571 \times 10$
	3	.3241	.3240	.3226	.3170	.3145	.3142	.3142	.3142
	4	.6183	.6180	.6148	.5745	.4911	.4733	.4715	.4713
	5	.6383	.6380	.6356	.6299	.6285	.6283	.6283	.6283
	6	.9325	.9320	.9269	.8698	.7977	.7867	.7855	.7854
	7	.9524	.9520	.9488	.9435	.9426	.9425	.9425	.9425
	8	$.1247 \times 10^{2}$	$.1246 \times 10^{2}$	$.1239 \times 10^{2}$	$.1169 \times 10^{2}$	$.1108 \times 10^{2}$	$.1100 \times 10^{2}$	$.1100 \times 10^{2}$	$.1100 \times 10^{2}$
	9	.1267	.1266	.1262	.1257	.1257	.1257	.1257	.1257

TABLE I.- ROOTS, γ_n, OF CHARACTERISTIC EQUATION (16) - Continued

(d) $\xi = 1,000$

ρ	n	γ_n for ζ = :							
		0.0001	0.001	0.01	0.1	1.0	10.0	100.0	1,000.0
0.01	1	0.4918×10^{-2}	0.4913×10^{-2}	0.4870×10^{-2}	0.4478×10^{-2}	0.2703×10^{-2}	0.9786×10^{-3}	0.3141×10^{-3}	0.9949×10^{-4}
	2	$.1475 \times 10^{-1}$	$.1474 \times 10^{-1}$	$.1461 \times 10^{-1}$	$.1349 \times 10^{-1}$	$.1073 \times 10^{-1}$	$.9936 \times 10^{-2}$	$.9846 \times 10^{-2}$	$.9837 \times 10^{-2}$
	3	.2459	.2457	.2435	.2265	.2016	$.1972 \times 10^{-1}$	$.1968 \times 10^{-1}$	$.1967 \times 10^{-1}$
	4	.3442	.3439	.3409	.3195	.2984	.2954	.2951	.2951
	5	.4426	.4422	.4383	.4139	.3960	.3937	.3935	.3935
	6	.5410	.5405	.5358	.5092	.4939	.4920	.4919	.4919
	7	.6394	.6388	.6332	.6053	.5919	.5904	.5903	.5902
	8	.7378	.7371	.7307	.7019	.6901	.6888	.6886	.6886
	9	.8362	.8354	.8282	.7989	.7883	.7872	.7871	.7870
0.1	1	0.1429×10^{-1}	0.1428×10^{-1}	0.1416×10^{-1}	0.1310×10^{-1}	0.8095×10^{-2}	0.2961×10^{-2}	0.9517×10^{-3}	0.3015×10^{-3}
	2	.4305	.4302	.4266	.3958	$.3145 \times 10^{-1}$	$.2894 \times 10^{-1}$	$.2866 \times 10^{-1}$	$.2863 \times 10^{-1}$
	3	.7227	.7221	.7160	.6676	.5916	.5776	.5762	.5760
	4	$.1020 \times 10^{0}$	$.1019 \times 10^{0}$	$.1010 \times 10^{0}$	.9477	.8814	.8718	.8708	.8707
	5	.1321	.1320	.1309	$.1235 \times 10^{0}$	$.1178 \times 10^{0}$	$.1171 \times 10^{0}$	$.1170 \times 10^{0}$	$.1170 \times 10^{0}$
	6	.1625	.1624	.1610	.1529	.1480	.1474	.1473	.1473
	7	.1932	.1930	.1914	.1828	.1784	.1779	.1779	.1779
	8	.2240	.2238	.2219	.2129	.2091	.2087	.2086	.2086
	9	.2550	.2547	.2526	.2433	.2399	.2395	.2395	.2395
1.0	1	0.2720×10^{-1}	0.2719×10^{-1}	0.2703×10^{-1}	0.2560×10^{-1}	0.1758×10^{-1}	0.6872×10^{-2}	0.2230×10^{-2}	0.7069×10^{-3}
	2	$.1083 \times 10^{0}$	$.1082 \times 10^{0}$	$.1073 \times 10^{0}$	.9942	.7491	$.6542 \times 10^{-1}$	$.6427 \times 10^{-1}$	$.6415 \times 10^{-1}$
	3	.2035	.2033	.2015	$.1870 \times 10^{0}$	$.1612 \times 10^{0}$	$.1559 \times 10^{0}$	$.1554 \times 10^{0}$	$.1553 \times 10^{0}$
	4	.3012	.3009	.2983	.2787	.2561	.2526	.2523	.2522
	5	.3997	.3994	.3958	.3725	.3532	.3507	.3505	.3504
	6	.4985	.4981	.4937	.4678	.4513	.4494	.4492	.4491
	7	.5975	.5969	.5917	.5641	.5499	.5482	.5481	.5480
	8	.6965	.6959	.6898	.6611	.6486	.6472	.6471	.6421
	9	.7957	.7950	.7880	.7586	.7475	.7463	.7462	.7462
10.0	1	0.3110×10^{-1}	0.3108×10^{-1}	0.3094×10^{-1}	0.2961×10^{-1}	0.2173×10^{-1}	0.9127×10^{-2}	0.3001×10^{-2}	0.9530×10^{-3}
	2	$.3172 \times 10^{0}$	$.3169 \times 10^{0}$	$.3141 \times 10^{0}$	$.2894 \times 10^{0}$	$.2069 \times 10^{0}$	$.1689 \times 10^{0}$	$.1637 \times 10^{0}$	$.1632 \times 10^{0}$
	3	.6296	.6291	.6235	.5775	.4931	.4753	.4734	.4732
	4	.9431	.9425	.9339	.8716	.7988	.7877	.7865	.7864
	5	$.1257 \times 10$	$.1256 \times 10$	$.1245 \times 10$	$.1171 \times 10$	$.1109 \times 10$	$.1101 \times 10$	$.1100 \times 10$	$.1100 \times 10$
	6	.1571	.1569	.1555	.1473	.1421	.1415	.1414	.1414
	7	.1884	.1883	.1866	.1779	.1733	.1728	.1728	.1728
	8	.2198	.2196	.2177	.2086	.2046	.2042	.2042	.2042
	9	.2512	.2509	.2488	.2395	.2359	.2356	.2355	.2355
100.0	1	0.3156×10^{-1}	0.3155×10^{-1}	0.3141×10^{-1}	0.3009×10^{-1}	0.2229×10^{-1}	0.9491×10^{-2}	0.3131×10^{-2}	0.9946×10^{-3}
	2	$.9940 \times 10^{0}$	$.9931 \times 10^{0}$	$.9843 \times 10^{0}$	$.9060 \times 10^{0}$	$.6427 \times 10^{0}$	$.5178 \times 10^{0}$	$.5006 \times 10^{0}$	$.4988 \times 10^{0}$
	3	$.1986 \times 10$	$.1984 \times 10$	$.1967 \times 10$	$.1821 \times 10$	$.1554 \times 10$	$.1497 \times 10$	$.1491 \times 10$	$.1490 \times 10$
	4	.2974	.2972	.2946	.2752	.2522	.2486	.2483	.2482
	5	.3147	.3147	.3146	.3142	.3140	.3140	.3140	.3140
	6	.3974	.3971	.3936	.3702	.3508	.3483	.3480	.3480
	7	.4967	.4962	.4918	.4659	.4493	.4473	.4471	.4471
	8	.5957	.5952	.5900	.5625	.5481	.5465	.5463	.5463
	9	.6285	.6285	.6284	.6281	.6279	.6278	.6278	.6278

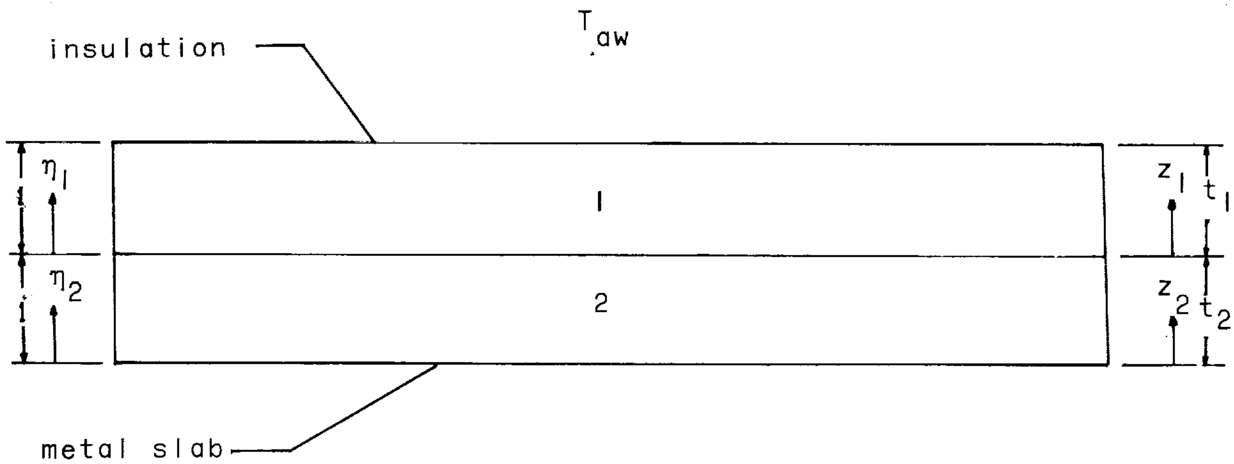
TABLE I.- ROOTS, γ_n, OF CHARACTERISTIC EQUATION (16) - Continued

(e) ξ = 10,000

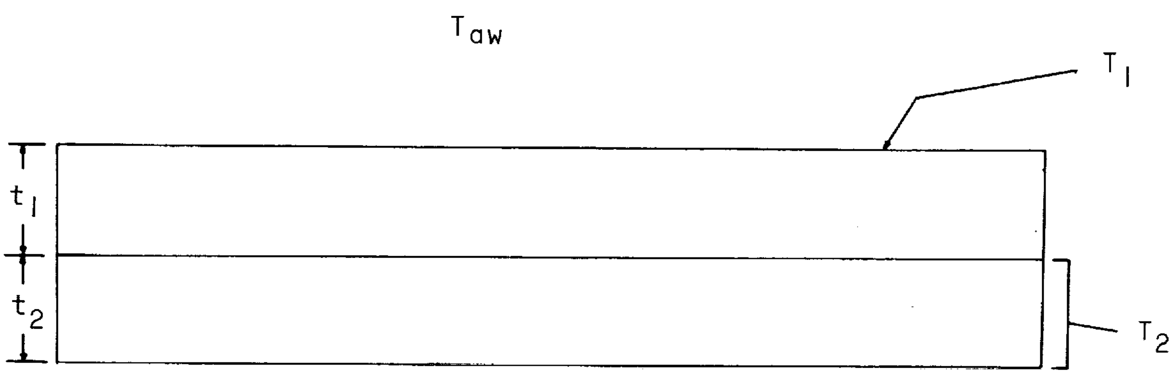
ρ	n	γ_n for ζ = : 0.0001	0.001	0.01	0.1	1.0	10.0	100.0	1,000.0
0.01	1	0.1555×10^{-2}	0.1554×10^{-2}	0.1540×10^{-2}	0.1416×10^{-2}	0.8549×10^{-3}	0.3095×10^{-3}	0.9934×10^{-4}	0.3146×10^{-4}
	2	.4665	.4661	.4620	.4267	$.3394 \times 10^{-2}$	$.3142 \times 10^{-2}$	$.3114 \times 10^{-2}$	$.3111 \times 10^{-2}$
	3	.7776	.7769	.7700	.7161	.6375	.6237	.6223	.6221
	4	$.1089 \times 10^{-1}$	$.1088 \times 10^{-1}$	$.1078 \times 10^{-1}$	$.1010 \times 10^{-1}$	.9436	.9342	.9333	.9332
	5	.1400	.1398	.1386	.1309	$.1252 \times 10^{-1}$	$.1245 \times 10^{-1}$	$.1244 \times 10^{-1}$	$.1244 \times 10^{-1}$
	6	.1711	.1709	.1694	.1610	.1562	.1556	.1555	.1555
	7	.2022	.2020	.2003	.1914	.1872	.1867	.1867	.1867
	8	.2333	.2331	.2311	.2220	.2182	.2178	.2178	.2178
	9	.2644	.2642	.2619	.2526	.2493	.2489	.2489	.2489
0.1	1	0.4518×10^{-2}	0.4514×10^{-2}	0.4478×10^{-2}	0.4143×10^{-2}	0.2560×10^{-2}	0.9363×10^{-3}	0.3010×10^{-3}	0.9533×10^{-4}
	2	$.1361 \times 10^{-1}$	$.1360 \times 10^{-1}$	$.1349 \times 10^{-1}$	$.1252 \times 10^{-1}$	.9945	$.9153 \times 10^{-2}$	$.9063 \times 10^{-2}$	$.9054 \times 10^{-2}$
	3	.2286	.2284	.2265	.2111	$.1871 \times 10^{-1}$	$.1827 \times 10^{-1}$	$.1822 \times 10^{-1}$	$.1822 \times 10^{-1}$
	4	.3225	.3222	.3195	.2997	.2788	.2757	.2754	.2754
	5	.4178	.4175	.4139	.3907	.3726	.3703	.3701	.3701
	6	.5141	.5137	.5092	.4837	.4680	.4661	.4659	.4659
	7	.6111	.6106	.6053	.5781	.5643	.5628	.5626	.5626
	8	.7086	.7080	.7019	.6735	.6613	.6600	.6599	.6599
	9	.8065	.8058	.7988	.7697	.7589	.7577	.7576	.7576
1.0	1	0.8603×10^{-2}	0.8598×10^{-2}	0.8549×10^{-2}	0.8095×10^{-2}	0.5560×10^{-2}	0.2173×10^{-2}	0.7051×10^{-3}	0.2235×10^{-3}
	2	$.3425 \times 10^{-1}$	$.3422 \times 10^{-1}$	$.3394 \times 10^{-1}$	$.3145 \times 10^{-1}$	$.2369 \times 10^{-1}$	$.2069 \times 10^{-1}$	$.2033 \times 10^{-1}$	$.2029 \times 10^{-1}$
	3	.6436	.6431	.6375	.5916	.5100	.4933	.4915	.4913
	4	.9528	.9520	.9436	.8815	.8099	.7991	.7980	.7979
	5	$.1264 \times 10^{0}$	$.1263 \times 10^{0}$	$.1252 \times 10^{0}$	$.1178 \times 10^{0}$	$.1117 \times 10^{0}$	$.1109 \times 10^{0}$	$.1109 \times 10^{0}$	$.1109 \times 10^{0}$
	6	.1577	.1576	.1562	.1480	.1428	.1421	.1421	.1421
	7	.1890	.1888	.1872	.1784	.1739	.1734	.1734	.1734
	8	.2203	.2201	.2182	.2091	.2052	.2047	.2047	.2047
	9	.2517	.2515	.2493	.2400	.2365	.2361	.2360	.2360
10.0	1	0.9836×10^{-2}	0.9831×10^{-2}	0.9786×10^{-2}	0.9363×10^{-2}	0.6872×10^{-2}	0.2886×10^{-2}	0.9491×10^{-3}	0.3014×10^{-3}
	2	$.1003 \times 10^{0}$	$.1002 \times 10^{0}$	$.9936 \times 10^{-1}$	$.9153 \times 10^{-1}$	$.6543 \times 10^{-1}$	$.5341 \times 10^{-1}$	$.5179 \times 10^{-1}$	$.5163 \times 10^{-1}$
	3	.1992	.1990	$.1972 \times 10^{0}$	$.1827 \times 10^{0}$	$.1560 \times 10^{0}$	$.1503 \times 10^{0}$	$.1497 \times 10^{0}$	$.1497 \times 10^{0}$
	4	.2983	.2981	.2954	.2757	.2527	.2492	.2488	.2488
	5	.3976	.3972	.3937	.3703	.3508	.3483	.3480	.3480
	6	.4969	.4964	.4920	.4661	.4495	.4475	.4473	.4473
	7	.5962	.5956	.5904	.5628	.5484	.5468	.5466	.5466
	8	.6955	.6949	.6888	.6600	.6474	.6460	.6459	.6459
	9	.7948	.7941	.7872	.7577	.7465	.7453	.7452	.7452
100.0	1	0.9983×10^{-2}	0.9978×10^{-2}	0.9933×10^{-2}	0.9517×10^{-2}	0.7050×10^{-2}	0.3001×10^{-2}	0.9902×10^{-3}	0.3145×10^{-3}
	2	$.3144 \times 10^{0}$	$.3142 \times 10^{0}$	$.3114 \times 10^{0}$	$.2866 \times 10^{0}$	$.2033 \times 10^{0}$	$.1638 \times 10^{0}$	$.1583 \times 10^{0}$	$.1578 \times 10^{0}$
	3	.6284	.6278	.6222	.5762	.4915	.4735	.4716	.4715
	4	.9425	.9416	.9332	.8709	.7980	.7868	.7856	.7855
	5	$.1257 \times 10$	$.1255 \times 10$	$.1244 \times 10$	$.1170 \times 10$	$.1109 \times 10$	$.1101 \times 10$	$.1100 \times 10$	$.1100 \times 10$
	6	.1571	.1569	.1555	.1473	.1421	.1414	.1414	.1414
	7	.1885	.1883	.1866	.1779	.1734	.1728	.1728	.1728
	8	.2199	.2197	.2178	.2087	.2047	.2042	.2042	.2042
	9	.2513	.2511	.2489	.2396	.2360	.2357	.2356	.2356

TABLE II.- ROOTS, γ_n, OF CHARACTERISTIC EQUATION (25) FOR $\zeta = 0$

ρ	n	γ_n for ξ = :						
		1.0	10.0	100.0	1,000.0	10,000.0	100,000.0	1,000,000.0
0.01	1	0.1555×10^{0}	0.4918×10^{-1}	0.1555×10^{-1}	0.4918×10^{-2}	0.1555×10^{-2}	0.4918×10^{-3}	0.1555×10^{-3}
	2	.4662	$.1475 \times 10^{0}$	.4666	$.1475 \times 10^{-1}$	.4666	$.1475 \times 10^{-2}$	.4666
	3	.7758	.2458	.7776	.2459	.7776	.2459	.7776
	4	$.1081 \times 10$	.3441	$.1089 \times 10^{0}$	.3442	$.1089 \times 10^{-1}$	.3442	$.1089 \times 10^{-2}$
	5	.1369	.4425	.1387	.4426	.1387	.4426	.1387
	6	.1571	.5404	.1711	.5411	.1711	.5411	.1711
	7	.1773	.6384	.2022	.6394	.2022	.6394	.2022
	8	.2060	.7362	.2333	.7372	.2333	.7372	.2333
	9	.2366	.8340	.2644	.8359	.2644	.8359	.2644
0.1	1	0.4486×10^{0}	0.1428×10^{0}	0.4524×10^{-1}	0.1429×10^{-1}	0.4523×10^{-2}	0.1429×10^{-2}	0.4523×10^{-3}
	2	$.1249 \times 10$	.4286	$.1361 \times 10^{0}$	.4306	$.1361 \times 10^{-1}$	.4306	$.1361 \times 10^{-2}$
	3	.1791	.7146	.2281	.7225	.2284	.7226	.2284
	4	.2550	.9996	.3221	$.1020 \times 10^{0}$	.3230	$.1019 \times 10^{-1}$	.3230
	5	.3446	$.1285 \times 10$	.4169	.1320	.4178	.1320	.4178
	6	.4279	.1571	.5128	.1626	.5140	.1626	.5140
	7	.4843	.1856	.6095	.1933	.6112	.1933	.6112
	8	.5551	.2142	.7066	.2251	.7087	.2250	.7087
	9	.6441	.2426	.8041	.2549	.8067	.2550	.8067
1.0	1	0.7854×10^{0}	0.2720×10^{0}	0.8590×10^{-1}	0.2720×10^{-1}	0.8590×10^{-2}	0.2720×10^{-2}	0.8590×10^{-3}
	2	$.2356 \times 10$	$.1051 \times 10$	$.3416 \times 10^{0}$	$.1083 \times 10^{0}$	$.3426 \times 10^{-1}$	$.1083 \times 10^{-1}$	$.3426 \times 10^{-2}$
	3	.3927	.1947	.6416	.2035	.6438	.2036	.6438
	4	.5498	.2768	.9496	.3013	.9529	.3014	.9529
	5	.7069	.3315	$.1260 \times 10$	.3998	$.1265 \times 10^{0}$	.3999	$.1264 \times 10^{-1}$
	6	.8369	.4051	.1571	.4986	.1577	.4988	.1577
	7	$.1021 \times 10^{2}$	.4943	.1882	.5976	.1890	.5978	.1890
	8	.1178	.5788	.2192	.6967	.2204	.6969	.2204
	9	.1335	.6344	.2500	.7946	.2513	.7948	.2513
10.0	1	0.8538×10^{0}	0.3063×10^{0}	0.9820×10^{-1}	0.3130×10^{-1}	0.9848×10^{-2}	0.3111×10^{-2}	0.9850×10^{-3}
	2	$.3321 \times 10$	$.2835 \times 10$	$.1000 \times 10$	$.3172 \times 10^{0}$	$.1004 \times 10^{0}$	$.3173 \times 10^{-1}$	$.1004 \times 10^{-1}$
	3	.6159	.3448	.1985	.6297	.1992	.6299	.1992
	4	.8748	.5977	.2934	.9432	.2984	.9435	.2984
	5	$.1049 \times 10^{2}$	.6590	.3184	$.1257 \times 10$	.3977	$.1257 \times 10^{0}$	.3977
	6	.1281	.9119	.3983	.1571	.4970	.1571	.4970
	7	.1563	.9731	.4965	.1885	.5963	.1886	.5963
	8	.1832	$.1226 \times 10^{2}$	.5934	.2198	.6956	.2200	.6956
	9	.2015	.1287	.6300	.2509	.7939	.2511	.7940
100.0	1	0.8595×10^{0}	0.3106×10^{0}	0.9960×10^{-1}	0.3160×10^{-1}	0.1000×10^{-1}	0.3160×10^{-2}	0.1000×10^{-2}
	2	$.3416 \times 10$	$.3162 \times 10$	$.3042 \times 10$	$.9943 \times 10^{0}$	$.3145 \times 10^{0}$	$.9946 \times 10^{-1}$	$.3145 \times 10^{-1}$
	3	.6416	.6269	.3241	$.1987 \times 10$	.6285	$.1988 \times 10^{0}$	.6285
	4	.9496	.9276	.6184	.2975	.9426	.2981	.9426
	5	$.1260 \times 10^{2}$	$.1007 \times 10^{2}$	.6383	.3147	$.1257 \times 10$	.3975	$.1257 \times 10^{0}$
	6	.1571	.1260	.9325	.3975	.1571	.4968	.1571
	7	.1882	.1570	.9524	.4968	.1885	.5962	.1885
	8	.2192	.1876	$.1247 \times 10^{2}$	.5958	.2199	.6955	.2199
	9	.2500	.1993	.1267	.6285	.2510	.7939	.2510



(a) Exact solution.



(b) Approximate solutions.

Figure 1.- Configuration and coordinate system of insulated panel.

NASA TN D-519
National Aeronautics and Space Administration.
AN ANALYSIS OF EXACT AND APPROXIMATE EQUATIONS FOR THE TEMPERATURE DISTRIBUTION IN AN INSULATED THICK SKIN SUBJECTED TO AERODYNAMIC HEATING. Robert S. Harris, Jr., and John R. Davidson. January 1961. 33p. OTS price, $1.00. (NASA TECHNICAL NOTE D-519)

Exact and approximate solutions are obtained, and the results are compared to determine the ranges of applicability of the approximations. The approximations are found to be within 5 percent of the exact solution when the ratio of the thermal capacitance of the metal to that of the insulation and the ratio of the conductance of the metal to that of the insulation are sufficiently large. The roots of the characteristic equation of the exact solution are generally applicable to the two-slab heat-transfer problem and are tabulated up to the first nine roots.

Copies obtainable from NASA, Washington

I. Harris, Robert S., Jr.
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